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# Moduli spaces of Riemann $N=1$ and $N=2$ supersurfaces

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This paper contains a full description of the moduli space of Riemann supersurfaces of types $N=1$ and $N=2$ (where $N$ is the number of supervariables). We describe topological invariants of supersurfaces, which determine the connected components of moduli space. Each connected component is represented as $T/\text{Mod}$, where $T \cong \mathbb{R}^{(n|m)}/(Z_2)^3$ and Mod is a discrete group.

*Keywords:* *supersurface, moduli space, Arf-functions, Fuchsian group, spinor bundles*
*1991 MSC: 14, 57, 81*

*Contents*



## Introduction

In the 1980s Riemann $N=1$ supersurfaces were presented for consideration in connection with the theory of superstrings [BS, Fr]. $N=2$ supersurfaces also appear in the theory [Ge]. The groups of uniformization for $N=2$ supersurfaces are described in [Ma]. Some local properties and especially the dimension of the moduli space of $N=2$ Riemann supersurfaces were found [FR].

The goal of this paper is a description of the connected components of the moduli space of Riemann $N=2$ supersurfaces. To this end we connect to each supersurface two Arf-functions. Below we describe topological types of pairs of Arf-functions and define the topological type of a Riemann $N=2$ supersurface as the topological type of its pair of Arf-functions. Then we prove that the set of all

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supersurfaces of the same topological type forms a connected space $M$, which is represented as $T/\mathrm{Mod}$, where $T \cong \mathbb{R}^{(n|m)}/(\mathbb{Z}_2)^3$ and Mod is a discrete group. We describe a construction connecting the system of generators of a super-Fuchsian group with $t \subset T \subset \mathbb{R}^{(n|m)}$. The construction also includes the moduli space of $N=1$ supersurfaces into the moduli space of $N=2$ supersurfaces. We prove that the body of the moduli space of $N=n$ supersurfaces is a space of spinor bundles of rank $n$. For $n=1$ this space coincides with the space of liftings of Fuchsian groups from $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ to $\tilde{\Gamma} \subset \mathrm{SL}(2, \mathbb{R})$. Here and in the following, when we say "Fuchsian group" we consider a "Fuchsian group whose tree acts on the upper half-plane".

## 1. Riemann $N=1$ and $N=2$ supersurfaces

Let $L=L(K)$ be the Grassmanian algebra over the field $K$ generated by the infinite set $1, l_1, l_2, \ldots$ . Each element $a \in L$ may be represented as the sum of an element $a_0 \in K$ and a finite number of elements of the form $a_{i_1 i_2 \cdots i_p} l_{i_1} \wedge l_{i_2} \wedge \cdots \wedge l_{i_p}$, where $p>0$, $a_{i_1 i_2 \cdots i_p} \in K$, and $i_1 > i_2 > \cdots > i_p$. The mapping $a \mapsto a^{\#} = a_0$ defines an epimorphism $\#: L \to K$. The monomial $l_{i_1} \wedge \cdots \wedge l_{i_p}$ is called *even* or *odd* depending on the parity of $p$. The monomial 1 is considered to be even. Linear combinations of even (odd) monomials with coefficients in the field $K$ form the set $L_0(K)$ of *even* elements (the set $L_1(K)$ of *odd* elements) of the algebra $L(K)$. The field $K$ will be assumed to be the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$.

Let $K^{(n|m)}$ be the linear superspace of all sequences $(z_1, \ldots, z_n | \theta_1, \ldots, \theta_m)$, where $z_i \in L_0(K)$ and $\theta_i \in L_1(K)$. Let $\#(z_1, \ldots, z_n | \theta_1, \ldots, \theta_m) = (z_1^{\#}, \ldots, z_n^{\#}) \subset K^n$.

Riemann $N=1$ supersurfaces with a non-commutative fundamental group are constructed by the $N=1$ *upper super-half-plane* $H' = \{(z, \theta) \in \mathbb{C}^{(1|1)} | \operatorname{Im} z^{\#} > 0\}$ and its group of automorphisms $\mathrm{Aut}(H')$, which consists of $A = I'\{a, b, c, d, e | \alpha, \beta\}$,

$$A(z|\theta) = \left( \frac{az+b+\gamma\theta}{cz+d+\delta\theta} \middle| \frac{e\theta+\alpha z+\beta}{cz+d+\delta\theta} \right),$$

where $a, b, c, d, e \in L_0(R)$, $\alpha, \beta, \gamma, \delta \in L_1(R)$, $ad-bc-\alpha\beta = e^2 + 2\gamma\delta$, $\alpha e = a\delta - c\gamma$, $\beta e = b\delta - d\gamma$ [BS]. The projection $\#: H' \to H^{\#}$ gives

$$A^{\#} = \#(A) \in \mathrm{Aut}(H^{\#}), \quad A^{\#} z = \frac{a^{\#} z + b^{\#}}{c^{\#} z + d^{\#}}.$$

We say that $\Gamma \subset \mathrm{Aut}(H')$ is an $N=1$ *super-Fuchsian group* if $\Gamma^{\#}$ is a Fuchsian group and $\#: \Gamma \to \Gamma^{\#}$ is the isomorphism. A *Riemann* $N=1$ *supersurface* is defined as $P = H'/\Gamma$, where $\Gamma$ is an $N=1$ super-Fuchsian group and $\Gamma^{\#}$ acts on $H^{\#}$ without fixed points.

The superspace $H=\{(z|\theta_1,\theta_2)\in\mathbb{C}^{(1|2)}\,|\,\mathrm{Im}\,z^{\#}>0\}$ is called the $N=2$ *upper super-half-plane*. Let $\mathrm{Aut}(H)$ be a *group of automorphisms* $A=I\{a,b,c,d,l\,|\,\epsilon\}$, which have the form

$$A(z|\theta_1,\theta_2)$$
$$=\left(\frac{az+b+\delta^{11}\theta_1+\delta^{12}\theta_2}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2}\,\middle|\,\frac{l^{11}\theta_1+l^{12}\theta_2+\epsilon^{11}z+\epsilon^{12}}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2},\frac{l^{21}\theta_1+l^{22}\theta_2+\epsilon^{21}z+\epsilon^{22}}{cz+d+\delta^{21}\theta_1+\delta^{22}\theta_2}\right),$$

where $a,b,c,d\in L_0(\mathbb{R})$, $l\in\mathrm{GL}(2,L_0(\mathbb{R}))$, $\epsilon,\delta\in\mathrm{GL}(2,L_1(\mathbb{R}))$,

$$\begin{pmatrix}-c & a\\ -d & b\end{pmatrix}\begin{pmatrix}\delta^{11} & \delta^{12}\\ \delta^{21} & \delta^{22}\end{pmatrix}=\begin{pmatrix}\epsilon^{21} & \epsilon^{11}\\ \epsilon^{22} & \epsilon^{12}\end{pmatrix}\begin{pmatrix}l^{11} & l^{12}\\ l^{21} & l^{22}\end{pmatrix}.$$

Moreover [Ma],

$$ad-bc-\epsilon^{11}\epsilon^{22}-\epsilon^{21}\epsilon^{12}=l^{11}l^{22}+l^{21}l^{12}+\delta^{11}\delta^{22}+\delta^{12}\delta^{21}=\Delta\,,\quad \Delta^{\#}>0\,,$$
$$l^{11}l^{21}+\delta^{11}\delta^{21}=l^{12}l^{22}+\delta^{12}\delta^{22}=0\,. \qquad (*)$$

The automorphism $A$ is called *hyperbolic* (*parabolic*) if $|a^{\#}+d^{\#}|>2$ ($|a^{\#}+d^{\#}|=2$). Each hyperbolic $A$ is conjugated to an automorphism

$$(z|\theta_1,\theta_2)\mapsto(\lambda z|h^{11}\theta_1+h^{12}\theta_2,h^{21}\theta_1+h^{22}\theta_2)\,,$$

where $\lambda^{\#}>1$, and each parabolic $A$ to

$$(z|\theta_1,\theta_2)\mapsto(z+1|h^{11}\theta_1+h^{12}\theta_2,h^{21}\theta_1+h^{22}\theta_2)\,.$$

From ($*$) it follows that $(h^{11}h^{22})^{\#}$, $(h^{12}h^{21})^{\#}\geqslant0$ and either $(h^{11})^{\#}=(h^{22})^{\#}=0$ or $(h^{12})^{\#}=(h^{21})^{\#}=0$. We define *functions* $\Omega_1,\Omega_2:\mathrm{Aut}_0(H)\to\{0,1\}$ *on the set* $\mathrm{Aut}_0(H)$ *of hyperbolic and parabolic automorphisms*, such that

(1) $\Omega_1(A)=1,\Omega_2(A)=1$ if $(h^{11})^{\#}$, $(h^{22})^{\#}>0$, $(h^{12})^{\#}=(h^{21})^{\#}=0$;
(2) $\Omega_1(A)=0,\Omega_2(A)=0$ if $(h^{11})^{\#}$, $(h^{22})^{\#}<0$, $(h^{12})^{\#}=(h^{21})^{\#}=0$;
(3) $\Omega_1(A)=1,\Omega_2(A)=0$ if $(h^{11})^{\#}=(h^{22})^{\#}=0$, $(h^{12})^{\#}$, $(h^{21})^{\#}>0$;
(4) $\Omega_1(A)=0,\Omega_2(A)=1$ if $(h^{11})^{\#}=(h^{22})^{\#}=0$, $(h^{12})^{\#}$, $(h^{21})^{\#}<0$.

The projection $\#:H\to H^{\#}$ gives $A^{\#}=\#(A)\in\mathrm{Aut}(H^{\#})$, where

$$A^{\#}z=\frac{a^{\#}z+b^{\#}}{c^{\#}z+d^{\#}}\,.$$

We say that $\Gamma\subset\mathrm{Aut}(H)$ is a $N=2$ *super-Fuchsian group* if $\Gamma^{\#}$ is a Fuchsian group and $\#:\Gamma\to\Gamma^{\#}$ is the isomorphism. A *Riemann* $N=2$ *supersurface* is defined as $P=H/\Gamma$, where $\Gamma\subset\mathrm{Aut}_0(H)$ is an $N=2$ super-Fuchsian group. The projection $\#:H\to H^{\#}$ gives the projection $\#:P\to P^{\#}$ on the Riemann surface $P^{\#}=H^{\#}/\Gamma$ (*body of* $P$).

## 2. Arf-functions connected with supersurfaces

In this section we prove that $\Omega_1$ and $\Omega_2$ generate two Arf-functions on the body $P^{\#}$ of a Riemann $N=2$ supersurface $P$.

**Lemma 2.1.** *Let* $\hat{H}=H^{\#}\times\mathbb{C}=\{(z|\theta)\}$ *and* $C_1$, $C_2$, $C_3\in\mathrm{Aut}(\hat{H})$ *be such that* $C_1C_2C_3=1$, $C_1(z|\theta)=(\lambda_1 z|\sigma_1\sqrt{\lambda_1}\theta)$,

$$C_2(z|\theta)=\left(\frac{(\lambda_2\alpha_2-\beta_2)z+(1-\lambda_2)\alpha_2\beta_2}{(\lambda_2-1)z+(\alpha_2-\lambda_2\beta_2)}\,\middle|\,\frac{\sigma_2\sqrt{\lambda_2}(\alpha_2-\beta_2)}{(\lambda_2-1)z+(\alpha_2-\lambda_2\beta_2)}\theta\right),$$

$$C_3(z|\theta)=\left(\frac{(\lambda_3\alpha_3-\beta_3)z+(1-\lambda_3)\alpha_3\beta_3}{(\lambda_3-1)z+(\alpha_3-\lambda_3\beta_3)}\,\middle|\,\frac{\sigma_3\sqrt{\lambda_3}(\alpha_3-\beta_3)}{(\lambda_3-1)z+(\alpha_3-\lambda_3\beta_3)}\theta\right),$$

*where* $\alpha_i$, $\beta_i$, $\lambda_i\in\mathbb{R}$, $\sigma=\pm1$, $\lambda_i>1$, $\alpha_2>0$. *Let* $l(C_i)\subset H^{\#}$ *be the half-circle with origin* $\beta_i$ *and end* $\alpha_i$ $(\alpha_1=\infty, \beta_1=0)$.

*Then: (1) if* $\beta_2<0$, *then* $\beta_2<\alpha_3<0<\alpha_2<\beta_3$ *and* $\sigma_3=\sigma_1\sigma_2$; *(2) if* $0<\beta_2<\alpha_2$ *then* $0<\alpha_3<\beta_2<\alpha_2<\beta_3$, $C_2 l(C_3)\cap l(C_3)\neq\emptyset$ *and* $\sigma_3=\sigma_1\sigma_2$; *(3) if* $\beta_2>\alpha_2$ *then either* $\sigma_3=\sigma_1\sigma_2$, $\beta_3<\alpha_3<0$ *or* $\alpha_2<\beta_3<\alpha_3<\beta_2$, $\sigma_3=\sigma_1\sigma_2$, *or* $\beta_2<\alpha_3<\beta_3$ *and in the last case*

$$0<\left(\frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{1+\sqrt{\lambda_1\lambda_2}}\right)^2\beta_2<\alpha_2<\beta_2<\alpha_3<\beta_3<\infty,$$

$$\sigma_3=-\sigma_1\sigma_2.$$

*Proof.* Let $C(0,\theta)=(f_0(C)+f_1(C)\theta, \psi_0(C)\theta+\psi_1(C))$. Then

$$\frac{\sigma_3\sqrt{\lambda_3}(\alpha_3-\beta_3)}{(\lambda_3-1)\alpha_3\beta_3}=\frac{\psi_0(C_3^{-1})}{f_0(C_3^{-1})}=\frac{\psi_0(C_1C_2)}{f_0(C_1C_2)}=\frac{\sigma_1\sigma_2\sqrt{\lambda_1\lambda_2}(\alpha_2-\beta_2)}{\lambda_1(1-\lambda_2)\alpha_2\beta_2},$$

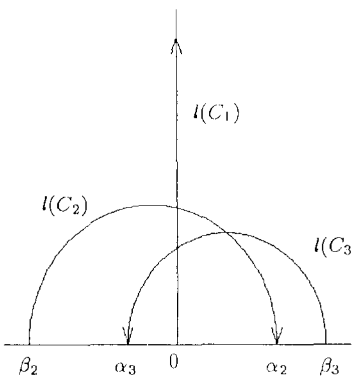


Fig. 1.

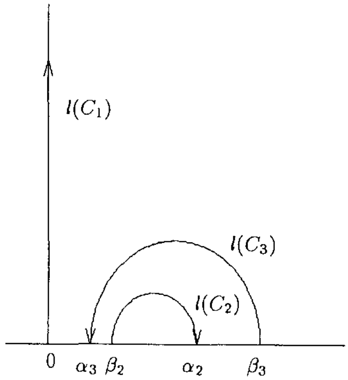


Fig. 2.

and therefore $\sigma_3 = \text{sign}\{(\beta_3 - \alpha_3)(\alpha_2 - \beta_2)\alpha_2\beta_2\alpha_3\beta_3\}\sigma_1\sigma_2$. Let $\beta_2 < 0$ (fig. 1). Then $C_3([\beta_2, 0]) = C_2^{-1}C_1^{-1}([\beta_2, 0]) \subset \beta_2, 0]$ and therefore $\beta_2 < \alpha_3 < 0$. Similarly it can be proved that $\beta_3 > \alpha_2$. Let $0 < \beta_2 < \alpha_2$ (fig. 2). Then $C_3([0, \beta_2]) = C_2^{-1}C_1^{-1}([0, \beta_2]) \subset [0, \beta_2]$ and therefore $0 < \alpha_3 < \beta_2$. Similarly $\beta_3 < \alpha_2$. Thus $C_2\alpha_3 = C_1^{-1}C_3^{-1}\alpha_3 \in [0, \alpha_3]$ and therefore $C_2\alpha_3 < \alpha_3$. Moreover, $\alpha_2 < C_2\beta_3 < \beta_3$ and therefore $C_2 l(C_3) \cap l(C_3) \neq \emptyset$. The statements (1) and (2) are proved.

Now let $\beta_2 > \alpha_2$ (fig. 3) and $D \in \text{Aut}(H)$ be such that $DC_2D^{-1} = I_{\lambda_2}^{\sigma_2}$. We put $\tilde{C}_1 = DC_2D^{-1}$, $\tilde{C}_2 = DC_3D^{-1}$, $\tilde{C}_3 = DC_1D^{-1}$. Then $l(\tilde{C}_1) \cap l(\tilde{C}_3) = \emptyset$. From statement (1) and the fact that $l(\tilde{C}_1) \cap l(\tilde{C}_3) = \emptyset$ it follows that $l(\tilde{C}_1) \cap l(\tilde{C}_2) = \emptyset$ and thus $l(C_2) \cap l(C_3) = \emptyset$. Similarly $l(C_1) \cap l(C_3) = \emptyset$. Thus all possible positions for $l(C_3)$ are shown in fig. 3. Now we use statement (2) for $\tilde{C}_1$, $\tilde{C}_2$ and obtain $\alpha_3 < \beta_3$ if $\alpha_3, \beta_3 > \beta_2$ and $\alpha_3 > \beta_3$, $\sigma_1\sigma_2\sigma_3 = 1$, if $\alpha_3, \beta_3 \in [\alpha_2, \beta_2]$ or $\alpha_3, \beta_3 < 0$.

The points $\alpha_3$ and $\beta_3$ are roots of the equation

$$(\lambda_2 - 1)x^2 - (\lambda_2\beta_2 - \alpha_2 - \lambda_1\beta_2 + \lambda_1\lambda_2\alpha_2)x + \lambda_1(\lambda_2 - 1)\alpha_2\beta_2 = 0 .$$

Therefore $\alpha_3\beta_3 = \lambda_1\alpha_2\beta_2 > \alpha_2\beta_2$ and the case $\alpha_3, \beta_3 \in [0, \alpha_2]$ is impossible. This

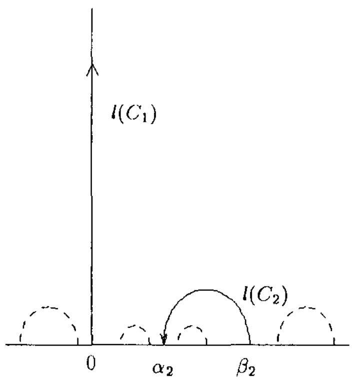


Fig. 3.

equation has real roots if and only if

$$\alpha_2 > \left(\frac{\sqrt{\lambda_1}+\sqrt{\lambda_1}}{1+\sqrt{\lambda_1\lambda_2}}\right)^2 \beta_2 \quad \text{or} \quad \alpha_2 < \left(\frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_1\lambda_2}-1}\right)^2 \beta_2 .$$

Since

$$\alpha_3+\beta_3 = \frac{\lambda_2\beta_2-\alpha_2-\lambda_1\beta_2+\lambda_1\lambda_2\alpha_2}{(\lambda_2-1)},$$

the last condition is incompatible with $\alpha_3, \beta_3 > \beta_2$. □

Let $\Gamma \subset \mathrm{Aut}_0(H)$ be an $N=2$ super-Fuchsian group, $P=H/\Gamma$, $p \in P^\#$, $\psi_\pi : \Gamma \to \pi_1(P^\#, p)$ and $\psi_H : \pi_1(P^\#, p) \to H_1(P^\#, Z_2)$ be the natural projections and $\Omega = \Omega_1 \psi_\pi^{-1} : \pi_1(P^\#, p) \to Z_2$.

**Lemma 2.2.** *Let the simple contours $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \subset P^\#$ be such that $\tilde{c}_i \cap \tilde{c}_j = p$ and let the corresponding $c_i \in \pi_1(P^\#, p)$ be such that $c_1c_2c_3=1$. Then $\Omega(c_3) = \Omega(c_1)+\Omega(c_2)+(c_1, c_2)$, where $(c_1, c_2) \in Z_2$ is the index intersection.*

*Proof.* Let $\psi_\pi^{-1}(c_i) = I\{a_i, b_i, e_i, d_i, l_i \mid \epsilon_i\}$ and $C_i \in \mathrm{Aut}(\hat{H})$,

$$C_i(z|\theta) = \left(\frac{a_i^\# z + b_i^\#}{e_i^\# z + d_i^\#} \,\middle|\, \frac{\mathrm{sign}(l^{11}+l^{12})^\# \Delta^\#}{e_i^\# z + d_i^\#}\theta\right).$$

Parabolic automorphisms are a limit of hyperbolic ones. Therefore the statement of the lemma for parabolic automorphisms follows from the statement for hyperbolic automorphisms. Let $C_1^\#, C_2^\#, C_3^\#$ be hyperbolic automorphisms. After conjugation we may assume that the $C_i$ satisfy lemma 2.1. The image of $l(C_i)$ on $P^\#$ is homotopic to $\tilde{c}_i$. Therefore $(c_1, c_2)=0$ if and only if $l(C_1) \cap Dl(C_2) = \emptyset$ for all $D \in \langle C_1, C_2 \rangle$ [Na1, lemma 1.4]. According to lemma 2.1 the last condition holds if and only if $\sigma_3 = -\sigma_1\sigma_2$, that is, if $\Omega(c_3) = \Omega(c_1) + \Omega_2(c_2)$. □

A set $F = \{a_i, b_i \ (i=1, ..., g), c_j \ (j=g+1, ..., g+n\}$ of generators of $\pi_1(P^\#, p)$ is called a *basis* of $P$ if the generators are represented by contours as on fig. 4, where the arrows show the orientation defined by multiplication on $i$ and $p$ is the unique point of intersection of the contours.

**Lemma 2.3.** *Let the simple contours $\tilde{a}_1, \tilde{a}'_1 \subset P^\#$ represent $a_1, a'_1 \in \pi_1(P^\#, p)$ such that $\psi_H(a_1) = \psi_H(a'_1)$. Then $\Omega(a_1) = \Omega(a'_1)$.*

*Proof.* Let us consider a basis $v = \{a_i, b_i \ (i=1, ..., g), c_j(j=g+1, ..., g+n)\} \subset \pi_1(P^\#, p)$ and $\psi_H v = \{\hat{a}_i, \hat{b}_i \ (i=1, ..., g), \hat{c}_j \ (j=g+1, ..., g+n)\} \subset H_1(P^\#, Z_2)$. We can take $\omega_v(\hat{a}_i) = \Omega(a_i)$, $\omega_v(\hat{b}_i) = \Omega(b_i)$, $\omega_v(\hat{c}_j) = \Omega(c_j)$ and continue $\omega_v$ on

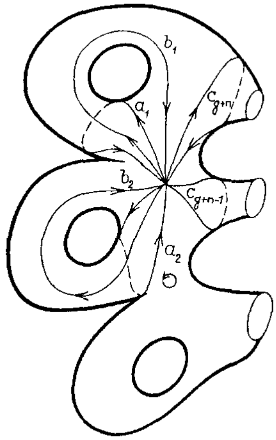


Fig. 4.

$H_1(P^{\#}, Z_2)$ by $\omega_v(d_1+d_2)=\omega_v(d_1)+\omega_v(d_2)+(d_1, d_2)$. According to lemma 2.2, for any Dehn twist $v\to sv$ [De] we have $\omega_{sv}=\omega_v$. Moreover, there exists a sequence of Dehn twists which maps $v$ to some $v'=\{a_i', b_i'\ (i=1, ..., g), c_j'\ (j=g+1, ..., g+n)$ [De]. Thus $\Omega(a_1')=\omega_{v'}(\hat{a}_1')=\omega_v(\hat{a}_1')=\omega_v(\hat{a}_1)=\Omega(a_1)$. □

Functions on simple contours $a\in\pi_1(P)$ which satisfy the statement of lemma 2.2. are known in topology as Arf-functions [DNF]. According to lemma 2.3, an Arf-function defines a function $\omega: H_1(P^{\#}, Z_2)\to Z_2$, which will be called an Arf-function too:

**Definition.** A function $\omega: H_1(P^{\#}, Z_2)\to Z_2$ is called an *Arf-function* if $\omega(a+b)=\omega(a)+\omega(b)+(a, b)$. (In [Na6] such functions are called spinor forms.)

**Definition.** Let $\Gamma\subset \mathrm{Aut}_0(H)$ be an $N=2$ super-Fuchsian group and $P=H/\Gamma$ be a Riemann $N=2$ supersurface. For $a\in H_1(P^{\#}, Z_2)$ we put $\omega_i(a)=\Omega_i(A)$, where $A\in\Gamma$ is such that $\psi_\pi(A)\in\pi_1(P^{\#}, p)$ may be represented by a simple contour and $\psi_H\psi_\pi(A)=a$.

**Theorem 2.1.** *The functions $\omega_1$, $\omega_2$ correctly define Arf-functions.*

*Proof.* The statement of the theorem for $\omega_1$ follows from lemmas 2.2, 2.3. Moreover, $\Omega_1+\Omega_2$ is a homomorphism and therefore $\omega_2$ is an Arf-function too. □

We shall say that the pair of Arf-functions $(\omega_1, \omega_2)$ is *generated* by the Riemann $N=2$ supersurface $P$.

## 3. Topological type of pairs of Arf-functions

Let $P$ be a surface of genus $g$ with $n$ holes and punctures. The system of generators $v=\{a_i, b_i\ (i=1, \ldots, g), c_j\ (j=g+1, \ldots, g+n)\}\in H_1(P, Z_2)$ is called a *basis* if $(a_i, a_j)=(b_i, b_j)=0$, $(a_i, b_j)=\delta_{ij}$ and each $c_j$ may be represented by contours which comprise one hole or puncture.

Each basis may be transformed into any other one by a sequence of transformations of the form: (1) $\tilde{a}_i=a_i+a_j$, $\tilde{b}_i=b_j$; $\tilde{a}_i=a_i$, $\tilde{b}_i=b_i+b_j$; (2) $\tilde{a}_i=a_i+b_i$; (3) $\tilde{a}_i=b_i$, $\tilde{b}_i=a_i$; (4) $\tilde{a}_i=a_i+c_i$; (5) $\tilde{c}_i=c_j$, $\tilde{c}_j=c_i$, where $1\leqslant i, j\leqslant g<k\leqslant g+n$ and $\tilde{a}_t=a_t$, $\tilde{b}_t=b_t$, $\tilde{c}_t=c_t$ if $t\neq i$, $t\neq j$ [DE].

Let $\omega: H_1(P, Z_2)\to Z_2$ be an Arf-function. Let us put $\delta=\delta(P, \omega)=0$ if there exists a basis $v$ such that

$$\sum_{i=1} \omega(a_i)\omega(b_i)\equiv 0 \pmod 2 .$$

In the other case we put $\delta=1$. The *type* of $\omega$ is $(g, \delta, n_0, n_1)$, where $n_\alpha=n_\alpha(P, \omega)$ is the number of such $j$ that $\omega(c_j)=\alpha$.

**Lemma 3.1.** *The set $(g, \delta, n_0, n_1)$ is the type of an Arf-function if and only if $n_1\equiv 0 \pmod 2$ and $\delta=0$ for $n_1>0$. In this case there exists a basis $v$ such that $\omega(a_i)=\omega(b_i)=0$ for $i>1$ and $\omega(a_1)=\omega(b_1)=\delta$.*

*Proof.* From

$$\sum_{j=g+1}^{g+n} \omega(c_j)=\omega\left(\sum_{j=g+1}^{g+n} c_j\right)=0$$

it follows that $n_1\equiv 0 \pmod 2$.

Now we shall prove that there exists a basis $v$ such that $\omega(a_i)=\omega(b_i)=0$ for $i>1$ and $\omega(a_1)=\omega(b_1)=0$ for $n_1>0$. To this end we shall transform to an arbitrary basis $v$. The transformations (1)–(3) give a basis $v$ such that $\omega(a_i)=\omega(b_i)=0$ for $i>1$ and $\omega(a_1)=\omega(b_1)$. Moreover, if $\omega(a_1)=\omega(b_1)=\omega(c_k)=1$ then for $\tilde{a}_1=a_1+c_k$, $\tilde{b}_1=b_1+c_k$ we have $\omega(\tilde{a}_1)=\omega(\tilde{b}_1)=0$. If $n_1=0$, then transformations (1)–(5) preserve the element $\sum_{i=1}^{g} \omega(a_i)\omega(\beta_i)\in Z_2$ and therefore the type $(g, 1, n_0, 0)$ exists. □

By the same method the following lemma may be proved.

**Lemma 3.2.** *Let $\chi: H_1(P, Z_2)\to Z_2$ be an epimorphism. Then there exists a basis $v$ such that $\chi(b_1)=\chi(a_i)=\chi(b_i)=0$ for $i>1$ and $\chi(a_1)=0$ if there exist such $c_k$ that $\chi(c_k)=1$.* □

Now let $\omega_1$, $\omega_2$ be two Arf-functions on $P$. We say that the set $(g, \delta_0(\omega_1, \omega_2), \delta(P, \omega_1), \delta(P, \omega_2), n_{\alpha\beta}, \alpha, \beta=0, 1)$ is the *type* of $(P, \omega_1, \omega_2)$ if $\delta_0=0$ for $\omega_1=\omega_2$, $\delta_0=1$ for $\omega_1 \not\equiv \omega_2$ and $n_{\alpha\beta}=n_{\alpha\beta}(P, \omega_1, \omega_2)$ is the number of elements $c_j$ of the basis $v$ such that $\omega_1(c_j)=\alpha$, $\omega_2(c_j)=\beta$.

**Lemma 3.3.** *The set $(g, 1, \delta_1, \delta_2, n_{\alpha\beta})$ is the type of a pair of Arf-functions $\omega_1 \not\equiv \omega_2$ if and only if the sets $(g, \delta_1, n_{00}+n_{01}, n_{10}+n_{11})$ and $(g, \delta_2, n_{00}+n_{10}, n_{01}+n_{11})$ are the types of some Arf-functions. In this case there exists a basis $v$ such that: (1) $\omega_j(a_i)=\omega_j(b_i)=0$ (for $j=1, 2$; $i>1$), $\omega_j(a_1)=\omega_j(b_1)=\delta_i$ if $n_{10}+n_{01}>0$; (2) $\omega_j(a_i)=\omega_j(b_i)=0$ (for $j=1, 2$; $i>1$), $\omega_1(b_1)=\omega_2(a_1)=\omega_2(b_1)=0$, $\omega_1(a_1)=1$ if $n_{10}=n_{01}=0$; $n_{11}>0$; (3) $\omega_j(a_i)=\omega_j(b_i)=0$ (for $j=1, 2$; $i>2$), $\omega_1(a_2)=\omega_1(b_2)=\omega_2(a_2)=\omega_2(b_2)=\epsilon$, $\omega_1(b_1)=\omega_2(b_1)=\delta$, $\omega_1(a_1)+\omega_2(a_1)=1$ (where $\epsilon=0$ if $\delta=1$ and $\omega_1(a_1)=1$ if $\delta=0$) if $n_{01}=n_{10}=n_{11}=0$.*

*Proof.* The first statement is obvious. Applying lemma 3.2 to $\omega=\omega_1+\omega_2$ we find a basis $v$ such that $\omega(b_1)=0$, $\omega(a_i)=\omega(b_i)=0$ for $i>1$, and $\omega(a_1)=0$ for $n_{01}+n_{10}>0$. Let $n_{01}+n_{10}>0$. Then $\omega_1(a_i)=\omega_2(a_i)$, $\omega_1(b_i)=\omega_2(b_i)$ and the transformations (1)–(3) give a basis $v$ such that $\omega_j(a_i)=\omega_j(b_i)=0$ for $j=1, 2$, $i>1$ and $\omega_1(a_1)=\omega_1(b_1)=\omega_2(a_2)=\omega_2(b_2)$. Now, using the transformations $\tilde{a}_1=a_1+c_k$ and $\tilde{b}_1=b_1+c_k$, we have $\omega_j(a_1)=\omega_j(b_1)=\delta_j$.

Now let $n_{01}=n_{10}=0$ and $b \in P$ be a simple contour which represents $b_1$. Then the functions $\omega_1$, $\omega_2$ are equal on $P \setminus b$ and according to lemma 3.1 there exists a basis $v$ of $H_1(P^{\#}, Z_2)$ such that $\omega_j(a_i)=\omega_j(b_i)=0$ for $j=1, 2$, $i>2$, $\omega_1(a_2)=\omega_1(b_2)=\omega_2(a_2)=\omega_2(b_2)=\epsilon$, $\omega_1(b_1)=\omega_2(b_1)=\delta$, $\omega_1(a_1)+\omega_2(a_1)=1$ and $\epsilon=0$ if $\delta=1$ or $n_{11}>0$. If $n_{11}>0$ then by the transformation $\tilde{b}_1=b_1+c_k$ we have $\delta=0$. If $\delta=0$, then the transformation $\tilde{a}_1=a_1+b_1$ gives $\omega_1(a_1)=1$. □

A basis which satisfies the conditions of lemma 3.1 for $\omega=\omega_1=\omega_2$ or of lemma 3.3. for $\omega_1 \not\equiv \omega_2$, is called a *basis adjusted to the pair* $(\omega_1, \omega_2)$ if $\omega_1(c_j)=\omega_2(c_j)=0$ for $i \leqslant g+n_{00}$; $\omega_1(c_j)=1$, $\omega_2(c_j)=0$ for $g+n_{00}<j \leqslant g+n_{00}+n_{01}$; $\omega_1(c_j)=0$, $\omega_2(c_j)=1$ for $g+n_{00}+n_{01}<j \leqslant g+n_{00}+n_{01}+n_{10}$; $\omega_1(c_j)=\omega_2(c_j)=1$ for $j>n_{00}+n_{01}+n_{10}$. It is obvious that these conditions define the value of $\omega_i$ on elements of the basis completely.

Let a set $(g, \delta_i, n_{\alpha\beta})$ satisfy the conditions of lemma 3.3 or have the form $(g, 0, \delta, \delta, n_0, 0, 0, n_1)$, where $(g, \delta, n_0, n_1)$ satisfies the conditions of lemma 3.1. Then we will say that the set $(g, \delta_i, n_{\alpha\beta})$ is *suitable*.

**Theorem 3.1.** *Let $(P_1, \omega_1^1, \omega_2^1)$ and $(P_2, \omega_1^2, \omega_2^2)$ be two pairs of Arf-functions, which have the same type. Then there exists a homeomorphism $\varphi: P_1 \to P_2$, which gives the isomorphism $\tilde{\varphi}: H_1(P_1, Z_2) \to H_1(P_2, Z_2)$ such that $\omega_j^2 \tilde{\varphi}=\omega_j^1$ for $j=1, 2$.*

*Proof.* According to lemmas 3.1 and 3.3 the groups $H_1(P_m, Z_2)$ have bases $v_m = \{a_i^m, b_i^m, c_j^m\}$ such that $\omega_j^1(a_i^1)=\omega_j^2(a_i^2)$, $\omega_j^1(b_i^1)=\omega_j^2(b_i^2)$, $\omega_j^1(c_i^1)=\omega_j^2(c_i^2)$. We construct $\varphi: P_1 \to P_2$ as the homeomorphism which moves $v_1$ to $v_2$.

## 4. Moduli of Riemann $N=1$ supersurfaces

For the first time moduli spaces of hyperbolic Riemann surfaces with arbitrary fundamental group were constructed by Fricke and Klein [FK]. Our approach to moduli spaces of supersurfaces is analogous to theirs. That is why we shall begin with a brief description of the Fricke–Klein theory. Its modern versions [Ke, Na2, Zi] differ by parameters which describe moduli spaces. An approach that is convenient for our goal can be found in [Na2], where the parameters describe generators of groups of uniformisation.

Each hyperbolic automorphism $A$ of the half-plane $H^{\#}$ (with $A(\infty) \neq \infty$) has the form $A=I(\lambda, \alpha, \beta)$,

$$Az = \frac{(\lambda\alpha-\beta)z+(1-\lambda)\alpha\beta}{(\lambda-1)z+(\alpha-\lambda\beta)},$$

where $\lambda, \alpha, \beta \in \mathbb{R}$, $\lambda > 1$, $\alpha \not\equiv \beta$. It is conjugated to $A=I(\lambda)$, $Az=\lambda z$ $(\lambda>1)$. We call the pair $(C_1=I(\lambda_1), C_2=I(\lambda_2, \alpha_2, \beta_2))$ *canonical* if

$$0<\left(\frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{1+\sqrt{\lambda_1\lambda_2}}\right)^2 \beta_2 < \alpha_2 < \beta_2 < \infty .$$

In this case $C_3=(C_1C_2)^{-1}=I(\lambda_3, \alpha_3, \beta_3)$ is hyperbolic and $\beta_2 < \alpha_3 < \beta_3 < \infty$ [Na2, proposition 2.1]. The pair $C_1$, $C_2$ is called *half-canonical* if there exists a $D \in \mathrm{Aut}(H^{\#})$ such that $(DC_1D^{-1}, DC_2D^{-1})$ is canonical. For a sequence $C_1, \ldots, C_n$ let us denote by $C^m$ the product $C_1 \cdots C_m$. A set $\{C_1, \ldots, C_n\} \subset \mathrm{Aut}(H^{\#})$ is called a *sequential set* if $C^n=1$, and $(C^m, C_{m+1})$ are half-canonical for all $m \leqslant n-2$.

We say that a pair $(A=I(\lambda_A, \alpha_A, \beta_A), B=I(\lambda_B, \alpha_B, \beta_B))$ *belongs to the automorphism* $I(\lambda)$ if

$$\infty \subset \alpha_A < \beta_B < \beta_A < \alpha_B < 0 ,$$

$$\frac{\alpha_A}{\beta_A} < \sqrt{\lambda}, \qquad \frac{\beta_B}{\alpha_B} < \sqrt{\lambda}, \qquad \lambda_B = \frac{\alpha_A\sqrt{\lambda}-\beta_A}{\beta_A\sqrt{\lambda}-\overline{\alpha_A}}, \qquad \lambda_A = \frac{\beta_B\sqrt{\lambda}-\alpha_B}{\overline{\alpha_B\sqrt{\lambda}}-\beta_B},$$

$$\alpha_B\beta_B\lambda - [(\alpha_A+\beta_A)(\alpha_B+\beta_B)-\alpha_B\beta_B-\alpha_A\beta_A]\sqrt{\lambda}+\alpha_A\beta_A=0 .$$

In this case $I(\lambda)=[A, B]$.

We say that a pair $(A, B)$ *belongs to a hyperbolic automorphism* $C$ if the pair $(DAD^{-1}, DBD^{-1})$ belongs to $I(\lambda)$ and $DCD^{-1}=I(\lambda)$ for some $D \in \mathrm{Aut}(H^{\#})$. Let $\tilde{Q}(C)$ be the set of all $(A, B)$ belonging to $C$. Let $Q(C)$ be the set of all sequences $(\alpha_A, \beta_A, \alpha_B, \beta_B)$ such that there exists $\lambda_A, \lambda_B$ for which $(I(\lambda_A, \alpha_A, \beta_A), I(\lambda_B, \alpha_B,$

$\beta_B))\in\tilde{Q}(C)$. It is obvious that $\tilde{Q}(C)\cong Q(C)\cong\mathbb{R}^3$.

Let $\tilde{T}(g, n)$ be the set of all sequences

$$\{\lambda_1, \alpha_1, \beta_1, ..., \lambda_{g+n}, \alpha_{g+n}, \beta_{g+n}, \alpha_{A_1}, \beta_{A_1}, \alpha_{B_1}, \beta_{B_1}, ..., \alpha_{A_g}, \beta_{A_g}, \alpha_{B_g}, \beta_{B_g}\}$$

such that

$$\{(C_1=I((\lambda_1, \alpha_1, \beta_1), ..., C_{g+n}=I(\lambda_{g+n}, \alpha_{g+n}, \beta_{g+n})\}$$

is a sequential set and

$$(\alpha_{A_i}, \beta_{A_i}, \alpha_{B_i}, \beta_{B_i})\in Q(C_i) .$$

It may be proved that $\tilde{T}(g, n)\cong\mathbb{R}^{6g+3n-3}$. The map $\alpha\mapsto D(\alpha)$, $\beta\mapsto D(\beta)$ defines the action of $\mathrm{Aut}(H^{\#})$ on $\tilde{T}(g, n)$. *The orbit space $T(g, n)\cong\mathbb{R}^{6g+3n-6}$ is called the Fricke–Klein space.*

Each point $\xi=\{\lambda_1, ..., \beta_{B_g}\}\in\tilde{T}(g, n)$ corresponds to the set

$$\Phi(\xi)=\{A_i, B_i\ (i=1, ..., g), C_j\ (j=g+1, ..., g+n)\} ,$$

where

$$C_i=I(\lambda_i, \alpha_i, \beta_i) , \qquad A_i=I(\lambda_{A_i}, \alpha_{A_i}, \beta_{A_i}) , \qquad B_i=I(\lambda_{B_i}, \alpha_{B_i}, \beta_{B_i})$$

and $(A_iB_i)$ belongs to $C_i$. They generate a Fuchsian group $\Gamma(\xi)$. If $\Gamma(\xi_1)=\Gamma(\xi_2)$, then $\Phi(\xi_1)=h\Phi(\xi_2)$, where $h$ is an automorphism of $\Gamma$. *Thus the moduli space $M(g, n)$ of Riemann surfaces of genus g with n holes is homeomorphic to $T(g, n)$/* $\mathrm{Mod}(g, n)$. It may be proved, that $\mathrm{Mod}(g, n)$ is isomorphic to the group of homotopic classes of homeomorphisms of the surface of genus $g$ with $n$ holes and $\mathrm{Mod}(n, g)$ acts discretely on $T(g, n)$.

The degenerations of some hyperbolic $C_i$ to parabolic automorphisms give the space $T(g, n, m)\cong\mathbb{R}^{6g+3n+2m-6}$ and the representation of the moduli space $M(g, n, m)$ of Riemann surfaces of genus $g$ with $n$ holes and $m$ punctures $(2g+n+m>2)$ in the form $M(g, n, m)=T(g, n, m)/\mathrm{Mod}(g, n, m)$, where $\mathrm{Mod}(g, n, m)\subset\mathrm{Mod}(g, n+m)$.

Let us now confine our attention to Riemann $N=1$ supersurfaces. They have the form $P=H'/\Gamma$, where $\Gamma$ is an $N=1$ super-Fuchsian group and each $A\subset\Gamma$ is hyperbolic or parabolic. Each hyperbolic map has the form

$$A=G'(\Omega_1|\lambda, \alpha, \beta|\epsilon^1, \epsilon^2)=I'\{\lambda\alpha-\beta, (1-\lambda)\alpha\beta, (\lambda-1), (\alpha-\lambda\beta), e|\epsilon^1, \epsilon^2\} ,$$

where $\alpha, \beta, \lambda, e\in L_0(\mathbb{R})$, $\epsilon^1, \epsilon^2\in L_1(\mathbb{R})$, $\lambda^{\#}>1$, $\Omega_1\in Z_2$ and $e$ is uniquely determined by the condition that $A$ is conjugated to a map $\tilde{A}(z|\theta)=(\tilde{\lambda}z|(2\Omega_1-1)\sqrt{\tilde{\lambda}}\theta)$. Each parabolic map is conjugated to some $\tilde{A}(z|\theta)=(z+1|(2\Omega_1-1)\theta)$, where $\Omega_1\in Z_2$. Thus we have the map $\Omega_1: \Gamma\to Z_2$.

This gives a function $\tilde{\omega}_1:\pi_1(P^{\#}, p)\to Z_2$ on the body $P^{\#}=H^{\#}/\Gamma^{\#}$ of the $N=1$ supersurface $P=H/\Gamma$. According to lemmas 2.2 and 2.3 an Arf-function $\omega_1:H_1(P^{\#}, Z_2)\to Z_2$ is generated. *The set $(g, \delta, n_\alpha, m_\alpha)$ is called the type of the*

$N=1$ *supersurface* $P$ if $(g, \delta, n_\alpha+m_\alpha)$ is the type of the form $\omega_1$ and $n_\alpha$ $(m_\alpha)$ is the number of holes (punctures) $c_j$ such that $\omega_1(c_j)=\alpha$. *Let* $M'(g, \delta, n_\alpha, m_\alpha)$ *be the space of all Riemann* $N=1$ *supersurfaces of type* $(g, \delta, n_\alpha, m_\alpha)$.

To investigate $M(g, \delta, n_\alpha. m_\alpha)$ consider a set

$$\tilde{T}'_*(g,n)=\tilde{T}'_1(g,n)\times\tilde{T}'_2(g,n)\times\tilde{T}'_3(g,n)\ ,$$

where

$$\tilde{T}'_1=\{\xi_1=(\lambda_{C_1},\alpha_{C_1},\beta_{C_1},\dots,\beta_{C_{g+n-1}},\alpha_{A_1},\beta_{A_1},\alpha_{B_1},\beta_{B_1},\dots,\beta_{B_g})$$
$$\in\mathbb{R}^{(7g+3n|0)}\,|\,\xi_1^\#\in\tilde{T}(g,n)\}\ ,$$
$$\tilde{T}'_2=\{\xi_2=(\epsilon^1_{A_1},\epsilon^2_{A_1},\epsilon^1_{B_1},\epsilon^2_{B_1},\dots,\epsilon^2_{B_g},\epsilon^1_{C_{g+1}},\epsilon^2_{C_{g+1}},\dots,\epsilon^2_{C_{g+n-1}})\in\mathbb{R}^{(0|4g+2n-4)}\}\ ,$$
$$\tilde{T}'_3=\{\xi_3=(\lambda_{A_1},\lambda_{B_1},\epsilon^1_{C_1},\epsilon^2_{C_1},\dots,\epsilon^2_{C_g})\in\mathbb{R}^{(2g|g)}\}\ .$$

Let $\chi=(g, \delta, n_\alpha)$ be the type of some Arf-function and $\xi=(\xi_1, \xi_2, \xi_3)$ $\in\tilde{T}'_*(g,n)$, where $n=n_0+n_1$. Then the set $\Phi(\xi_1^\#)=\{A_i^*, B_i^*\ (i=1, ..., g),\ C_j^*$ $(j=g+1, ..., g+n)\}$ generates a Fuchsian group $\Gamma(\xi_1^\#)$ and defines a basis

$$v=\{a_i, b_i\ (i=1,\dots,g), c_j\ (j=g+1,\dots,g+n)\}\subset H_1(P^*, Z_2)\ ,$$

where $P^*=H^\#/\Gamma(\xi_1)$.

There exists a unique Arf-function $\omega_1$ on $P$ for which $\omega_1(a_1)=\omega_1(b_1)=\delta$, $\omega_1(a_i)=\omega_1(b_i)=0$ for $i>1$ and $\omega_1(c_j)=1$ for $j>g+n_0$. Let us put

$$C_i=G(\omega_1(c_i)\,|\,\lambda_{C_i},\alpha_{C_i},\beta_{C_i}\,|\,\epsilon^1_{C_i},\epsilon^2_{C_i})\ (i<g+n)\ ,$$
$$C_{g+n}=(C_1\cdots C_{g+n-1})^{-1}\ ,$$
$$A_i=G(\omega_1(a_i)\,|\,\lambda_{A_i},\alpha_{A_i},\beta_{A_i}\,|\,\epsilon^1_{A_i},\epsilon^2_{A_i})\ ,$$
$$B_i=G(\omega_1(b_i)\,|\,\lambda_{B_i},\alpha_{B_i},\beta_{B_i}\,|\,\epsilon^1_{B_i},\epsilon^2_{B_i})\ .$$

Let $\tilde{T}'(\chi)=\{\xi\in\tilde{T}_*(g,n)\,|\,C_i=[A_iB_i]\ (i=1,\dots,g)\}$. The group $\mathrm{Aut}(H')$ acts on $\tilde{T}'(\chi)$ by conjugates. Let us put $T'(\chi)=\tilde{T}'(\chi)/\mathrm{Aut}(H')$. It may be proved that $T'(\chi)\cong\mathbb{R}^{(6g+3n-6|4g+2n-4)}/(Z_2)$. From this follows

**Theorem 4.1** [Na3, Na4]. *The space* $M'(\chi)=M'(g, \delta, n_\alpha, m_\alpha)$ *is not empty if and only if* $n_1+m_1\equiv0 \pmod 2$ *and* $\delta=0$ *for* $n_1+m_1>1$. *In this case* $M'(\chi)=T'(\chi)/\mathrm{Mod}(\chi)$, *where* $T'(\chi)=\mathbb{R}^{(6g+3n-6|4g+2(n+m)-4)}/(Z_2)$, $n=n_0+n_1$, $m=m_0+m_1$ *and* $\mathrm{Mod}(\chi)\subset\mathrm{Mod}(g,n,m)$. □

## 5. Moduli of Riemann $N=2$ supersurfaces

According to section 2 each Riemann $N=2$ supersurface $P$ generates a pair of Arf-functions $(\omega_1, \omega_2)$ on $P^\#$. The type $\chi=(g, \delta_i, n_{\alpha\beta})$ of such a pair was described in section 3. Now let us describe *the space $M(\chi)$ of all Riemann $N=2$ supersurfaces with a pair of Arf-functions of type $\chi$*. Each $M(\chi)$ will be represented in the form $\tilde{T}_*(g,n)/\tilde{\text{Mod}}(\chi)$, where

$$\tilde{T}_*(g,n)=\tilde{T}_1(g,n)\times\tilde{T}_2(g,n)\times\tilde{T}_3(g,n)\ ,$$

$$\tilde{T}_1=\{\xi_1=(\lambda_{C_1},\alpha_{C_1},\beta_{C_1},\dots,\beta_{C_{g+n-1}},\alpha_{A_1},\beta_{A_1},\alpha_{B_1},\beta_{B_1},\dots,\beta_{B_g}) \subset \mathbb{R}^{(7g+3n|0)}\,|\,\xi_1^\#\in\tilde{T}(g,n)\}\ ,$$

$$\tilde{T}_2(g,n)=\{\xi_2=(\gamma_{A_1},\epsilon_{A_1},\gamma_{B_1},\epsilon_{B_1},\dots,\epsilon_{B_g},\gamma_{C_{g+1}},\epsilon_{C_{g+1}},\dots,\epsilon_{C_{g+n-1}})\,|\, \gamma_{A_i},\gamma_{B_i},\gamma_{C_i}\in L_0(\mathbb{R}),\gamma_{A_i}^\#,\gamma_{B_i}^\#,\gamma_{C_i}^\#>0,\epsilon_{A_i},\epsilon_{B_i},\epsilon_{C_i}\in \mathrm{GL}(2,L_1(\mathbb{R}))\}\ ,$$

$$\tilde{T}_3(g,n)=\{\xi_3=(\lambda_{A_1},\lambda_{B_1},\gamma_{C_1},\epsilon_{C_1},\dots,\epsilon_{C_g})\,|\, \lambda_{A_i},\lambda_{B_i},\lambda_{C_i}\in L_0(\mathbb{R}),\lambda_{A_i}^\#,\lambda_{B_i}^\#>1,\gamma_{C_i}^\#>0,\epsilon_{C_i}\in \mathrm{GL}(2,L_1(\mathbb{R}))\}\ .$$

Each hyperbolic transformation $A$ may be represented in the form

$$A=G(\omega_1,\omega_2\,|\,\lambda,\alpha,\beta,\gamma\,|\,\epsilon) =I\{\lambda\alpha-\beta,\ (1-\lambda)\alpha\beta,\ (\lambda-1),\ (\alpha-\lambda\beta),\sqrt{\lambda}(\alpha-\beta)h\,|\,\epsilon\}\ ,$$

where $(\sqrt{\lambda})^2=\lambda$, $(\sqrt{\lambda})^\#>0$, $\gamma^\#>0$ and $h^{ij}$ are completely described by the conditions (1) $h^{11}=-\gamma$, $(h^{22})^\#<0$ if $\omega_1=\omega_2=0$; (2) $h^{12}=-\gamma$, $(h^{21})^\#<0$ if $\omega_1=0$, $\omega_2=1$; (3) $h^{12}=\gamma$, $(h^{21})^\#>0$ if $\omega_1=1$, $\omega_2=0$; (4) $h^{11}=\gamma$, $(h^{22})^\#>0$ if $\omega_1=\omega_2=1$.

Let $\chi=(g,\delta_i,n_{\alpha\beta})$ be a suitable type and $\xi=(\xi_1,\xi_2,\xi_3)\in\tilde{T}_*(g,n)$, where $n=n_{00}+n_{01}+n_{10}+n_{11}$. Then a set $\Phi(\xi_1^\#)=\{A_i^*,\ B_i^*\ (i=1,\dots,g),\ C_j^*\ (j=g+1,\dots,g+n)\}$ generates a Fuchsian group $\Gamma(\xi_1^\#)$ and defines a basis $v=\{a_i,b_i\ (i=1,\dots,g),\ c_j\ (j=g+1,\dots,g+n)\}\subset H_1(P^*,\mathrm{Z}_2)$, where $P^*=H^\#/\Gamma(\xi_1^\#)$. It is obvious that there exists a unique pair $(P^*,\omega_1,\omega_2)$ of Arf-functions of type $\chi$ for which $v$ is a basis adjusted to it.

Let us put

$$C_i=G(\omega_1(c_i),\omega_2(c_i)\,|\,\lambda_{C_i},\alpha_{C_i},\beta_{C_i},\gamma_{C_i}\,|\,\epsilon_{C_i})\ (i<g+n)\ ,$$

$$C_{g+n}=(C_1\cdots C_{g+n-1})^{-1}\ ,$$

$$A_i=G(\omega_1(a_i),\omega_2(a_i)\,|\,\lambda_{A_i},\alpha_{A_i},\beta_{A_i},\gamma_{A_i}\,|\,\epsilon_{A_i})\ ,$$

$$B_i=G(\omega_1(b_i),\omega_2(b_i)\,|\,\lambda_{B_i},\alpha_{B_i},\beta_{B_i},\gamma_{B_i}\,|\,\epsilon_{B_i})\ .$$

Let

$$\tilde{T}(\chi)=\{\xi\in\tilde{T}_*(g,n)\,|\,C_i=[A_i,B_i]\ (i=1,\dots,g)\}\ .$$

Each $\xi\in\tilde{T}(\chi)$ generates the transformations $V(\chi,\xi)=\{A_i,B_i,C_j\}$, which generate an $N=2$ super-Fuchsian group $\Gamma(\chi,\xi)$ and a Riemann $N=2$ supersurface $P(\chi,\xi)=H/\Gamma(\chi,\xi)$.

**Lemma 5.1.** *The correspondence* $\xi \mapsto P(\chi, \xi)$ *maps* $\tilde{T}(\chi)$ *on* $M(\chi)$.

*Proof.* The transformations $A_i$, $B_i$, $C_j$ were chosen in such a way that the supersurface $P(\chi, \xi)$ generates a pair of Arf-functions $(\omega_1, \omega_2)$. Thus $P(\chi, \xi) \in M(\chi)$. Now let $P \in M(\chi)$. It generates a pair of Arf-functions $(P^{\#}, \omega_1, \omega_2)$. Let $v = \{a_i, b_i, c_j\} \subset H_1(P^{\#}, Z_2)$ be a basis adjusted to $(\omega_1, \omega_2)$. Let us consider a basis $\tilde{v} = \{\tilde{a}_i, \tilde{b}_i, \tilde{c}_j\} \subset \pi_1(P^{\#}, p)$, which turn into $v$ by the natural projection $\pi_1(P^{\#}, p) \to H_1(P^{\#}, Z_2)$. Let $V = \{A_i, B_i, C_j\} = \psi_{\pi_1}^{-1}(\tilde{v})$. Then the set $\{\lambda_D, \alpha_D, \beta_D, \gamma_D, \epsilon_D | D \in v, D = G(\omega_1(D), \omega_2(D) | \lambda_D, \alpha_D, \beta_D, \gamma_D | \epsilon_D)\}$ forms a point $\xi \in \tilde{T}_*(g, n)$ such that $P = P(\chi, \xi)$. □

Different points $\xi$, $\xi' \in \tilde{T}(\chi)$ may give the same point of $M(\chi)$. This occurs in two cases: (1) $\Gamma(\xi') = FT(\xi)F^{-1}$, where $F \in \mathrm{Aut}(H)$; (2) $\Gamma(\chi, \xi) = \Gamma(\chi, \xi')$ and $V(\chi, \xi') = fV(\chi, \xi)$, where $f \in \mathrm{Aut}(\Gamma(\chi, \xi))$. Thus $M(\chi) = T(\chi)/\mathrm{Mod}(\chi)$, where $T(\chi) = \tilde{T}(\chi)/\mathrm{Aut}(H)$ and $\mathrm{Mod}(\chi) \subset \mathrm{Mod}(g, n)$.

**Lemma 5.2.** $T(\chi) = T_0(\chi)/G$, *where* $G = (Z_2)^3$, $T_0(\chi) \cong \mathbb{R}^{(8g+4n-8+b|8g+4n-8)}$, $n = n_{00} + n_{01} + n_{10} + n_{11}$ *and (1)* $b = 0$ *for* $\delta_0 = 1$; *(2)* $b = 1$ *for* $\delta_0 = 0$, $n > 0$; *(3)* $b = 2$ *for* $\delta_0 = n = 0$.

*Proof.* The natural projections $\tilde{T}_1 \times \tilde{T}_2 \times \tilde{T}_3 \to \tilde{T}_1 \times \tilde{T}_2$ give a one-to-one correspondence between $\tilde{T}(\chi) \subset \tilde{T}_1(g, n) \times \tilde{T}_2(g, n) \times \tilde{T}_3(g, n)$ and $\tilde{T}_1(\chi) \times \tilde{T}_2(\chi)$, where $\tilde{T}_i(\chi) \subset \tilde{T}_i(g, n)$. In each orbit of $\tilde{T}(\chi)$ under the action of $\mathrm{Aut}(H)$ there exists a $\xi \in \tilde{T}(\chi)$ such that $C_1(z | \theta_1, \theta_2) = (\lambda z | \theta_i, \theta_{3-i})$ and $\alpha_{A_1} = 1$ (or $\alpha_{C_2} = 1$ if $g = 0$). Let $\hat{T}(\chi) \subset \tilde{T}_1(\chi) \times \tilde{T}_2(\chi)$ be the set of such $\xi$. It has the form $\hat{T}_1(\xi) \times \hat{T}_2(\xi)$, where $\hat{T}_1(\xi)$ (parallel to the classical situation) is homeomorphic to $\mathbb{R}^{(6g+3n-6|0)}$ and $\hat{T}_2(\xi) = \mathbb{R}^{(2g+n-b|8g+4n-4)}/G$, where $G = \{(z | \theta_1, \theta_2) \mapsto (z | \pm\theta_i, \pm\theta_{3-i})\}$ and (1) $b = 0$ if $\delta_0 = 0$, $n = 0$; (2) $b = 1$ if $\delta_0 = 0$, $n > 0$; (3) $b = 1$ if $\delta_0 = 1$.

If $\delta_0 = 0$ then $T(\chi) = \hat{T}(\chi)$. If $\delta = 1$ then the conjugation $(z | \theta_1, \theta_2) \mapsto (z | \mu\theta_1, \mu^{-1}\theta_2)$ acts on $\hat{T}_2(\xi)$ and $T(\chi) = \hat{T}(\chi)/L_0(\mathbb{R}) \cong \mathbb{R}^{(8g+4n-8|8g+4n-8)}$. □

Now let us consider Riemann $N = 2$ supersurfaces with punctures. The type of the supersurface $P$ with holes and punctures has the form $(g, \delta_0, \delta_1, \delta_2, n_{\alpha\beta}, m_{\alpha\beta})$, where $(g, \delta_i, n_{\alpha\beta} + m_{\alpha\beta})$ is the type of the pair of Arf-functions $\omega_i : H_1(P^{\#}, Z_2) \to Z_2$ and $m_{\alpha\beta}$ is the number of such punctures $C_j$ that $\omega_1(C_j) = \alpha$, $\omega_2(C_j) = \beta$. A supersurface with punctures may be constructed by degeneration of part of the holes to punctures. This is equivalent to a degeneration of part of the hyperbolic transformations $C_j$ to parabolic ones, which depend on $(3|4)$ parameters. Thus the combination of lemmas 5.2, 3.1 and 3.3 gives

**Theorem 5.1** *Let* $M(\chi) = M(g, \delta_0, \delta_1, \delta_2, n_{\alpha\beta}, m_{\alpha\beta})$ *be the space of all Riemann* $N = 2$ *supersurfaces of the type* $\chi$. *Then* $M(\chi) \neq \emptyset$ *if and only if (1)* $n_{01} + m_{01} + n_{11} + m_{11}$, *and* $n_{10} + m_{10} + n_{11} + m_{11}$ *are even*; *(2)* $\delta_2 = 0$ *if*

$n_{01}+m_{01}+n_{11}+m_{11}>0$; *(3)* $\delta_1=0$ *if* $n_{10}+m_{10}+n_{11}+m_{11}>0$; *(4)* $n_{01}=n_{10}=m_{01}=m_{10}=0$ *if* $\delta_0=0$. *In this case* $M(\chi)\cong T(\chi)/\mathrm{Mod}(\chi)$, *where* $\mathrm{Mod}(\chi)\subset \mathrm{Mod}(g, n, m)$ *and* $T(\chi)=\mathbb{R}^{(8g+4n+3m-8+b|8g+4n+3m-8)}/G$, *where* $G=(Z_2)^3$, $n=n_{00}+n_{01}+n_{10}+n_{11}$, $m=m_{00}+m_{01}+m_{10}+m_{11}$, *and (1)* $b=0$ *if* $\delta_0=1$, *(2)* $b=1$ *if* $\delta_0=0$, $n+m>0$, *(3)* $b=2$ *if* $\delta_0=n=m=0$.

In the following we describe the inclusion of the moduli space of $N=1$ supersurfaces in the moduli space of $N=2$ supersurfaces.

Let $\sigma(z|\theta_1, \theta_2)=(z|\theta_2, \theta_1)$ and $\mathrm{Aut}^*(H)=\{A\in \mathrm{Aut}(H)\,|\,\sigma A=A\}$. Each $A\in \mathrm{Aut}^*(H)$ has the form $A=I\{a, b, c, d, l|\epsilon\}$, where $\epsilon^{11}=\epsilon^{21}$, $\epsilon^{12}=\epsilon^{22}$, $l^{11}=l^{22}$, $l^{12}=l^{21}$. Let us put $A'=\varphi(A)=I'\{a, b, c, d, l^{11}+l^{12}|\sqrt{2}\epsilon^{11}, \sqrt{2}\epsilon^{12}\}\in \mathrm{Aut}(H')$. For $A'=I'\{a, b, c, d, l|\alpha, \beta\}$ let us put $A=\varphi^0(A')=I'\{a, b, c, d, l|\epsilon\}\in \mathrm{Aut}^*(H)$, where $\epsilon^{11}=\epsilon^{21}=\alpha/\sqrt{2}$, $\epsilon^1=\epsilon^2=\beta/\sqrt{2}$,

$$l^{11}=l^{22}=e+\frac{e\alpha\beta}{2(ad-bc)}, \qquad l^{12}=l^{21}=-\frac{e\alpha\beta}{2(ad-bc)}.$$

A direct calculation gives

**Lemma 5.3.** *The map* $\varphi:\mathrm{Aut}^*(H')\to \mathrm{Aut}(H')$ *is an epimorphism and* $\mathrm{Ker}\,\varphi=\{1, \sigma\}$. *The map* $\varphi^0:\mathrm{Aut}(H')\to\mathrm{Aut}^*(H)$ *is a monomorphism,* $\varphi\varphi^0=1$ *and* $\mathrm{Aut}^*(H)\cong \mathrm{Im}\,\varphi^0\times \mathrm{Ker}\,\varphi$. *If* $A=G(\Omega_1, \Omega_2|\lambda, \alpha, \beta, \gamma|\epsilon)$ *then* $\varphi(A)=G'(\Omega_1|\lambda, \alpha, \beta|\sqrt{2}\epsilon^{11}, \sqrt{2}\epsilon^{12})$. *If* $A'=G'(\Omega|\lambda, \alpha, \beta|\epsilon^1, \epsilon^2)$ *then* $\varphi^0(A')=A=F(\Omega, E|\lambda, \alpha, \beta, \gamma|\epsilon)$, *where* $\gamma$ *and* $\epsilon$ *are determined uniquely from the condition* $\varphi(A)=A'$.

Thus if $P=H/\Gamma$ is an $N=2$ supersurface such that $\Gamma\subset \mathrm{Aut}^*(H)\cap \mathrm{Aut}_0(H)$, then $\Gamma'=\varphi(\Gamma)$ is an $N=1$ super-Fuchsian group and $\varphi(P)=H'/\Gamma'$ is a Riemann $N=1$ supersurface. Let now $M^*(\chi)\subset M(\chi)$ be the set of all $N=2$ supersurfaces $P=H/\Gamma\in M(\chi)$ such that $\Gamma\subset \mathrm{Aut}^*(H)$.

**Theorem 5.2.** *If* $\chi=(g, \delta_i, n_{\alpha\beta}, m_{\alpha\beta})$ *such that* $M(\chi)\neq\emptyset$ *then* $\varphi(M^*(\chi))=M'(\chi')$, where $\chi'=(g, \delta_1, n_{\alpha 0}+n_{\alpha 1}, m_{\alpha 0}+m_{\alpha 1})$. *The map* $\varphi:M^*(\chi)\to M'(\chi')$ *is a one-to-one correspondence if* $\delta_0=0$ *and a finite-sheet cover if* $\delta_0=1$.

*Proof.* Let $P=H/\Gamma\in M(\chi)$ and let $\omega_1, \omega_2:H_1(P^\#, Z_2)\to Z_2$ be Arf-functions generated by $P$. As before, the case $m_{\alpha\beta}>0$ follows from $m_{\alpha\beta}=0$ by passage to the limit. If $m_{\alpha\beta}=0$ then $\Gamma$ consists of hyperbolic transformations $A=G(\Omega_1(A), \Omega_2(A)|\lambda, \alpha, \beta, \gamma|\epsilon)$ and according to lemma 5.3 $\varphi(A)=G'(\Omega_1(A)|\lambda, \alpha, \beta|\sqrt{2}\epsilon^{11}, \sqrt{2}\epsilon^{12})$. Thus the Riemann $N=1$ supersurface $P'=\varphi(P)$ generates the Arf-function $\omega_1$ and therefore $P'\in M'(\chi')$.

If $\tilde{P}'=H'/\tilde{\Gamma}'\in M'(\chi)$, then according to lemma 5.3, $H/\varphi^0(\tilde{\Gamma}')\in M^*(\chi)$ and thus $\varphi(M^*(\chi))=M'(\chi')$. Moreover, if $\tilde{P}=H/\tilde{\Gamma}\in M(g, 0, \delta, \delta, n_{\alpha\beta}, m_{\alpha\beta})$, then the transformation $A\in\tilde{\Gamma}$ is determined by $\varphi(A)$ and according to lemma 5.3 $\tilde{\Gamma}=\varphi^0(\varphi(\tilde{\Gamma}))$. Thus $\varphi: M^*(\chi)\to M'(\chi')$ is a one-to-one correspondence if $\delta_0=0$.

Let now $\Omega:\tilde{\Gamma}'\to\{1, \sigma\}$ be an epimorphism and $\varphi^\Omega:\Gamma'\to\mathrm{Aut}^*(H)$ have the form $\varphi^\Omega(A)=\varphi^0(A)\cdot\Omega(A)$. If $\delta_0=1$ we can select $\Omega$ such that $H/\Gamma^\Omega(\tilde{\Gamma}')\in M^*(\chi)$ and therefore $\varphi(M^*(\chi))=M'(\chi')$. Moreover, if $\varphi(\tilde{P})=\tilde{P}'$ and $\tilde{P}\in M^*(\chi)$, then according lemma 5.3 $\tilde{P}=H/\varphi^\Omega(\tilde{\Gamma}')$ for some $\Omega$. □

## 6. Liftings of Fuchsian groups, spinor bundles and the body of the moduli space of Riemann supersurfaces

The set $F$ of all $N=2$ super-Fuchsian groups $\Gamma\subset\mathrm{Aut}(H)$ contains a subset $F_{\mathbb{R}}$ which consists of the transformations $A=I\{a, b, c, d, l|0\}$, where $a, b, c, d\in\mathbb{R}$, $l\in\mathrm{GL}(2, \mathbb{R})$. By our definition the classes of conjugation of $F$ form a moduli space $M$ of $N=2$ supersurfaces. Let $M_{\mathbb{R}}$ be a subset which corresponds to $F_{\mathbb{R}}$. The projection $\#: L(\mathbb{R})\to\mathbb{R}$ gives the projections $\#_F: F\to F_{\mathbb{R}}$ and $\#_M: M\to M_{\mathbb{R}}$. For each $m\in M$ the set $(\#_M)^{-1}(m)$ is $\tilde{\mathbb{R}}\backslash\tilde{\mathbb{R}}^\#$, where $\tilde{\mathbb{R}}\cong\mathbb{R}^{(n_1|n_2)}$. Thus $M_{\mathbb{R}}$ is the body of $M$.

Let us give a more detailed description of $M_{\mathbb{R}}$. It may be seen that each $\Gamma\subset F_{\mathbb{R}}$ acts on $\check{H}=H\times\mathbb{C}^2=\{(z|\theta_1, \theta_2)\}$ by

$$A(z|\theta_1, \theta_2)=\left(\frac{az+b}{cz+d}\,\middle|\,\frac{l^{11}\theta_1+l^{12}\theta_2}{cz+d}, \frac{l^{21}\theta_1+l^{22}\theta_2}{cz+d}\right)$$

for each $A=I\{a, b, c, d, l|0\}$. Let $P=\check{H}/\Gamma$. Then the natural projection $\check{H}\to H$ gives a bundle $f_\Gamma: P\to P^\#$ of rank 2.

**Theorem 6.1** [Na5]. *The bundles $f_\Gamma$ are spinor bundles.*

*Proof.* Consider a scalar product

$$((z_0|\theta_1, \theta_2)(z_0|\theta_1, \theta_2))=\theta_1\theta_2+\theta_2\theta_1$$

on $z_0\times\mathbb{C}^2\subset\check{H}$. Then

$$(A(z_0|\theta_1, \theta_2), A(z_0|\theta_1, \theta_2))=(l^{11}l^{22}+l^{12}l^{21})(\theta_1\theta_2'+\theta_2\theta_1')/(cz_0+d)^2$$

$$=((z_0|\theta_1, \theta_2), (z_0|\theta_1, \theta_2))\frac{\mathrm{d}}{\mathrm{d}z}A^\#(z)\ .$$

Thus the scalar product on $z_0\times\mathbb{C}^2$ gives a scalar product $(\cdot,\cdot)$ on $f_\Gamma^{-1}(p_0)$, which depends on the choice of a local map $\epsilon: u\to\mathbb{C}$, $p_0\in u$ and is changed to $(\cdot,\cdot)\,\mathrm{d}\tilde{\epsilon}/\mathrm{d}\epsilon$ by the change $\tilde{\epsilon}=\tilde{\epsilon}(\epsilon)$. Therefore $f_\Gamma$ is a spinor bundle in the sense of [Mu]. □

The inclusion of the moduli space $M'$ of $N=1$ supersurfaces into $M$ gives an inclusion of the body $M'_{\mathbb{R}}$ of $M'$ into $M_{\mathbb{R}}$. This body is the set of classes of conjugation of a set $F'_{\mathbb{R}} \subset F_{\mathbb{R}}$. For $\Gamma \subset F'_{\mathbb{R}}$, the $\Gamma \subset \mathrm{Aut}^*(H)$ and the bundle $f_\Gamma$ is $f'_\Gamma \oplus f'_\Gamma$, where $f'_\Gamma$ is a spinor bundle of rank 1.

This bundle has another description. It may be considered that the group $\Gamma \subset F'_{\mathbb{R}}$ acts on $\hat{H} = H^\# \times \mathbb{C} = \{(z|\theta)\}$ as an $N=1$ super-Fuchsian group, that is,

$$A(z|\theta) = \left(\frac{az+b}{cz+d} \,\middle|\, \frac{l^{11}\theta}{cz+d}\right),$$

where $A = I\{a, b, c, d, l|0\} \in \Gamma$. Then the natural projection $\hat{H} \to H^\#$ gives the bundle $f'_\Gamma : \hat{H}/\Gamma \to P^\#$.

**Theorem 6.2.** *The correspondence $\Gamma \to f'_\Gamma$ gives a bijection between $M'_{\mathbb{R}}$ and the set of all spinor bundles of rank* 1.

*Proof.* According to theorem 6.1 $f'_\Gamma$ is a spinor bundle. Let $f: E \to P$ be a spinor bundle of rank 1. Then a uniformization of $P$ gives a uniformization of $E$ and a representation $E = \hat{H}/\Gamma$, where $\Gamma$ consists of the transformations

$$A(z|\theta) = \left(\frac{az+b}{cz+d} \,\middle|\, \frac{l\theta}{cz+d}\right).$$

Thus $\Gamma$ is an $N=1$ super-Fuchsian group. □

Let $\Gamma \subset F'_{\mathbb{R}}$ be an $N=1$ super-Fuchsian group. For $A \subset \Gamma$,

$$A(z|\theta) = \left(\frac{az+b}{cz+d} \,\middle|\, \frac{l\theta}{cz+d}\right),$$

put

$$G(A) = \begin{pmatrix} a/l & b/l \\ c/l & d/l \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

**Lemma 6.1** [Na6]. *The map $G: \Gamma \to \mathrm{SL}(2, \mathbb{R})$ is a monomorphism for each $\Gamma \in F'_{\mathbb{R}}$.*

*Proof.* We will prove that $G(C_1C_2) = G(C_1)G(C_2)$, where $C_1$, $C_2$ and $C_3 = C_2^{-1}C_1^{-1}$ are hyperbolic transformations. Obviously, $G(A^{-1}) = (G(A))^{-1}$ and $G(ACA^{-1}) = G(A)G(C)G(A^{-1})$. Therefore it is possible to assume that $C_1$, $C_2$, $C_3$ are as in lemma 2.1. Then

$$G(C_1)=\frac{\sigma_1}{\sqrt{\lambda_1}}\begin{pmatrix}\lambda_1 & 0\\ 0 & 1\end{pmatrix},\qquad G(C_i)=\frac{\sigma_1}{\sqrt{\lambda_i}\,(\alpha_i-\beta_i)}\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix},$$

where $a_i=\lambda_i\alpha_i-\beta_i$, $b_i=(1-\lambda_i)\alpha_i\beta_i$, $c_i=\lambda_i-1$, $d_i=\alpha_i-\lambda_i\beta_i$ $(i=2,3)$. Thus

$$\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}=G(C_1)G(C_2)G(C_3)=\frac{\sigma_1\sigma_2\sigma_3}{\sqrt{\lambda_1\lambda_2\lambda_3}\,(\alpha_2-\beta_2)(\alpha_3-\beta_3)}\begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

where $b=c=c_2a_3+d_2c_3=0$ and $a=d=\lambda_1(a_2a_3+b_2c_3)$. Therefore

$$a_3=-(d_2/c_2)c_3\,,$$

$$a=d=\lambda_1c_3(b_2-a_2d_2/c_2)=-\lambda_1(c_3/c_2)\lambda_2(\alpha_2-\beta_2)^2<0\,,$$

$$\sigma=\sigma_1\sigma_2\sigma_3\,\mathrm{sign}(\alpha_2-\beta_2)\,\mathrm{sign}(\beta_3-\alpha_3)\,.$$

Using lemma 2.1 we have $\sigma=1$. □

Each $A=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{R})$ generates automorphisms

$$P(A)z=\frac{az\pm b}{cz+d}$$

of $H^{\#}$. A subgroup $\tilde{\Gamma}\subset\mathrm{SL}(2,\mathbb{R})$ will be called SL-Fuchsian if $\Gamma=P(\tilde{\Gamma})\subset\mathrm{Aut}_0(H^{\#})$ is a Fuchsian group and $\#|_{\tilde{\Gamma}}:\tilde{\Gamma}\to\Gamma$ is an isomorphism. According to [AAS], for each Fuchsian group $\Gamma\subset\mathrm{Aut}(H)$ there exists a *lifting* of $\Gamma$, that is, an SL-Fuchsian group $\tilde{\Gamma}$ such that $P(\tilde{\Gamma})=\Gamma$.

**Theorem 6.3.** *The correspondence $\Gamma\to G(\Gamma)$ gives a bijection between $M'_{\mathbb{R}}$ and the classes of conjugation of all SL-Fuchsian groups.*

*Proof.* Let $\tilde{\Gamma}$ be an SL-Fuchsian group. If $\tilde{A}=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\tilde{\Gamma}$ and $A=G^{-1}(\tilde{A})$, then

$$A(z|\theta)=\left(\frac{az+b}{cz+d}\,\middle|\,\frac{\theta}{cz+d}\right).$$

Let us prove that $G^{-1}(\tilde{\Gamma})\in F'_{\mathbb{R}}$ is a group. Let $\tilde{C}_1$, $\tilde{C}_2\in\tilde{\Gamma}$. Then $C_i=G^{-1}(\tilde{C}_i)$ generates a free group $\langle C_1, C_2\rangle$ and therefore $\langle C_1, C_2\rangle\in F'_{\mathbb{R}}$. According to lemma 6.1, $G(C_1C_2)=G(C_1)G(C_2)=\tilde{C}_1\tilde{C}_2$. Thus $G^{-1}(\tilde{C}_1\tilde{C}_2)=C_1C_2=G^{-1}(\tilde{C}_1)G^{-1}(\tilde{C}_2)$, $G^{-1}(\tilde{\Gamma})$ is group and therefore $G^{-1}\tilde{\Gamma})\in F'_{\mathbb{R}}$. □

Let us now describe the set of all SL-Fuchsian groups. Put

$$\Omega\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{cases}1 & \text{if } a+d>0\,,\\ 0 & \text{if } a+d<0\,.\end{cases}$$

If $|a+d|>2$, then

$$\left(\frac{az+b}{cz+d}\,\middle|\,\frac{\theta}{cz+d}\right)=\left(\frac{(\lambda\alpha-\beta)z+(1-\lambda)\alpha\beta}{(\lambda-1)z+(\alpha-\lambda\beta)}\,\middle|\,\frac{(2\Omega-1)(\alpha-\beta)\theta}{(\lambda-1)z+(\alpha-\lambda\beta)}\right),$$

where $\Omega=\Omega(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$. Thus according to lemmas 2.2 and 2.3 for an arbitrary SL-Fuchsian group $\tilde{\Gamma}$ the function $\Omega$ defines on the Riemann surface $P=H^{\#}/P(\tilde{\Gamma})$ an Arf-function $\omega_{\tilde{\Gamma}}:H_1(P, Z_2)\to Z_2$.

**Theorem 6.4.** *For an arbitrary Fuchsian group $\Gamma\subset \mathrm{Aut}_0(H^{\#})$ the map $\tilde{\Gamma}\mapsto\omega_{\tilde{\Gamma}}$ defines a bijection between liftings of $\Gamma$ and Arf-functions $\omega:H_1(P, Z_2)\to Z_2$ on $P=H^{\#}/\Gamma$.*

*Proof.* As proved in section 4, each Arf-function $\omega:H_1(P, Z_2)\to Z_2$ is generated with some $\Gamma'\in F'_{\mathbb{R}}$. Then according to lemma 6.1, $\tilde{\Gamma}=G(\Gamma')$ is an SL-Fuchsian group and $\omega_{\tilde{\Gamma}}=\omega$. □

The projection # gives a projection $\#:M'_{\mathbb{R}}(\chi)\to M^{\#}(g, n)$ of the body of the moduli space of $N=1$ Riemann supersurfaces of type $\chi=(g, \delta, n_0, n_1)$ on the moduli space of Riemann surfaces of type $(g, n_0+n_1)$. From theorems 4.1, 6.3 and 6.4 follows

**Theorem 6.5.** *The map $M'_{\mathbb{R}}(g, \delta, n_0, n_1)\to M^{\#}(g, n_0+n_1)$ covers $m$ sheets, where*

$$m=2^{2g+n_0+n_1-1} \quad \text{if } n_1>0\,,$$

$$m=2^{\max(n_0,1)+g-2}(2^g-1) \quad \text{if } n_1=0, \delta=1\,,$$

$$m=2^{\max(n_0,1)-2}(2^g+1) \quad \text{if } n_1=\delta=0\,.$$

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